Week 03 – problem set

Nonlinear Optics for Quantum Technologies

March 6th, 2025

1 The anharmonic (nonlinear) Lorentz oscillator model

1. By setting $\lambda = 0$ the equation 3 (excluding the term $m\beta_2 x(t)^2$ is solved, see lecture notes, with

$$x_1(t) = \frac{1}{2}\left(X_1(\omega)\exp(-i\omega t) + c.c.\right)$$

, where $X_1 = \frac{\frac{qE}{m}}{\omega_0^2 - \omega^2 - i\gamma\omega}$ and $\chi_1 = \frac{\epsilon_0 \frac{Nq^2}{m}}{\omega_0^2 - \omega^2 - i\gamma\omega}$. Inserting $x_1(t)$ into the equation and keeping only linear terms of λ we obtain:

$$\lambda\left[\ddot{x}_2 + \omega_0^2 x_2 + \gamma\dot{x}_2\right] = -\lambda\beta_2 x_1^2 = -\lambda\frac{\beta_2}{4}\left(X_1^2 e^{-i2\omega t} + 2X_1X_1^* + X_1^{*2}e^{i2\omega t}\right)$$

2. Two new frequencies appear: one at zero frequency (amplitude $0.5\beta_2|X_1|^2$) and one at frequency 2ω

3. This can now be solved for the two frequencies $x_2 = X_2(\omega = 0) + 0.5 * (X_2(2\omega)e^{-i2wt} + cc)$. As for $\omega = 0$ the time derivative is zero, we have that:

$$X_2(\omega = 0) = -\frac{-\beta|X_1|^2}{2\omega_0^2}$$

For $X_2(2\omega)$ we have instead a full equation only replacing the driving term $\frac{qE}{m}$ by $-\frac{\beta_2}{4}X_1^2$. We thus obtain

$$X_2(2\omega) = -\frac{\beta_2}{2}X_1(\omega)^2 X_1(2\omega)\frac{m}{qE}$$

4. The two relations follow directly from the solution of the previous step, knowing that $X_1 \propto \chi_1$.

2. Wave propagaⁿ in anisotropic media - Part II (1a)

1) • Maxwell eq
$$\begin{cases} \nabla \cdot D = 0 & \nabla \cdot B = 0 \\ \nabla \times E = -\dfrac{\partial B}{\partial t} & \nabla \times H = \dfrac{\partial D}{\partial t} \end{cases}$$

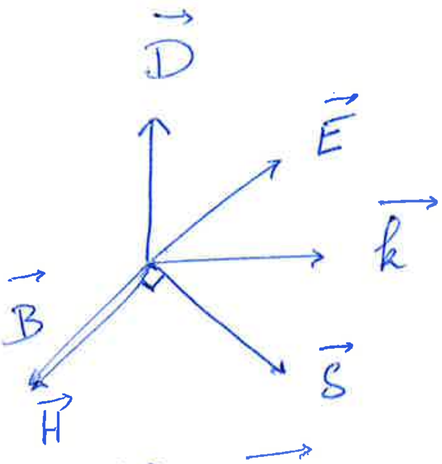
• Constitutives eq : $D = \varepsilon_0 \underline{\underline{\varepsilon}} E$

$$B = \mu_0 H$$

• Conversion to frequency domain

$$\nabla \cdot = i k \qquad \frac{\partial}{\partial t} = -i\omega$$

↳ Maxwell eq.
$$\begin{cases} k \cdot D = 0 & k \cdot B = 0 \\ k \times E = \mu_0 \omega H & k \times H = -\omega D \end{cases}$$



$\cdot \vec{S} = \vec{E} \times \vec{H}$

$$k \times H = -\omega D$$

$$k \times (k \times E) = -\mu_0 \omega^2 D$$
$$= -\frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} E$$

$\vec{k} \perp \vec{D}$

$\vec{S} \perp \vec{E}$

$$\frac{1}{c} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad k \times (k \times \hat{e}) = -\frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \hat{e}$$

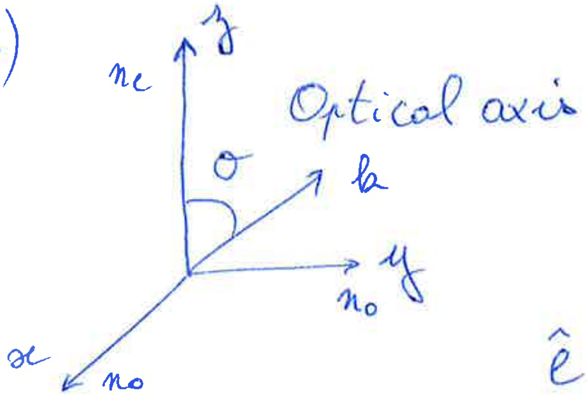
$$\Longrightarrow \quad k \times (k \times \hat{e}) + \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \hat{e} = 0$$

1a) In the isotropic case $\underline{\underline{\mathcal{E}}} = \mathcal{E}$ and $E \perp k$

$$\hookrightarrow -k^2 \hat{e} + \frac{\omega^2}{c^2} \mathcal{E} u = 0 \qquad (1b)$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \mathcal{E} .$$

2b)



Isotropic in (x, y) plane

Considering propagation at angle θ to the optical axis Oz

$$\hat{e} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \qquad k \begin{pmatrix} 0 \\ k \sin\theta \\ k\cos\theta \end{pmatrix}$$

$$k \times (k \times \hat{e}) = (k \cdot \hat{e}) k - k^2 \hat{e}$$

$$(\hookrightarrow a \times (b \times c) = (c \times b) \times a = b(a \cdot c) - c(a \cdot b))$$

$$= (k \sin\theta \, e_y + k \cos\theta \, e_z) k - k^2 \hat{e}$$

$$\begin{cases} Ox : -k^2(\sin^2\theta \, e_x + \cos^2\theta \, e_x) + \frac{\omega^2}{c^2} \underbrace{\mathcal{E}_{xx}}_{\mathcal{E}_\perp} e_x = 0 & (1) \\ Oy : k^2(\cos\theta \sin\theta \, e_z - \cos^2\theta \, e_y) + \frac{\omega^2}{c^2} \underbrace{\mathcal{E}_{yy}}_{\mathcal{E}_\perp} e_y = 0 & (2) \\ Oz : k^2(\sin^2\theta \, e_z + k^2 \cos\theta \sin\theta \, e_y) + \frac{\omega^2}{c^2} \underbrace{\mathcal{E}_{zz}}_{\mathcal{E}_\parallel} e_z = 0 & (3) \end{cases}$$

$$\underline{\underline{\mathcal{E}}} = \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}$$

(2a)

(1) $-k^2 e_x + \frac{\omega^2}{C^2} n_o^2 e_x = 0$

$\hookrightarrow$ 2 solutions : $\begin{cases} (a) & k = \frac{\omega}{c} n_o \\ (b) & e_x = 0 \end{cases}$

Fist solution (a)

(2) $(k\cos\theta\, e_z + k\sin\theta\, e_y)\, k\sin\theta - k^2 e_y + \underbrace{\frac{\omega^2}{C^2} n_o^2}_{k^2} e_y = 0$

(3) $(k\cos\theta\, e_z + k\sin\theta\, e_y)\, k\cos\theta - \underbrace{k^2 e_z + \frac{\omega^2}{C^2} n_e^2 e_z}_{\left(\frac{\omega^2}{C^2} n_e^2 - k^2\right) e_z} = 0$

$\hookrightarrow \begin{cases} (2) \Rightarrow k \cdot e = 0 \\ (3) \Rightarrow e_z = 0 \end{cases} \Rightarrow e_y = 0 \,,\ \hat{e} \parallel Ox$

Second solution (b)

(2) $\left(-k^2\cos^2\theta + \frac{\omega^2}{C^2} n_o^2\right) e_y + \left(k^2\cos\theta\sin\theta\right) e_z = 0$

(3) $\left(k^2\cos\theta\sin\theta\right) e_y + \left(-k^2\sin^2\theta + \frac{\omega^2}{C^2} n_e^2\right) e_z = 0$

$$k^4\cos^2\theta\sin^2\theta + \left(\frac{\omega}{c}\right)^4 n_o^2 n_e^2 - k^2\cos^2\theta\, \frac{\omega^2}{C^2} n_e^2$$
$$- k^2\sin^2\theta\, \frac{\omega^2}{C^2} n_o^2 - k^4\sin^2\theta\cos^2\theta = 0$$

$$k = \frac{\omega}{c} n_o$$

$$\left(\frac{\omega}{c}\right)^4 n_o^2 n_e^2 - n_\theta^2 \left(\frac{\omega}{c}\right)^4 \cos^2\theta \, n_e^2 - n_\theta^2 \left(\frac{\omega}{c}\right)^4 n_o^2 \sin^2\theta = 0 \qquad (2b)$$

$$\Rightarrow \frac{1}{n_\theta^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e}$$

One gets two orthogonal polarisation vectors

First case: Ordinary wave

$D_o \perp k$

$D_o \perp$ optical axis

$D_o \parallel E_o$

index : n_o

Second case : Extraordinary

$D_e \perp k$

$D_e \perp D_o$

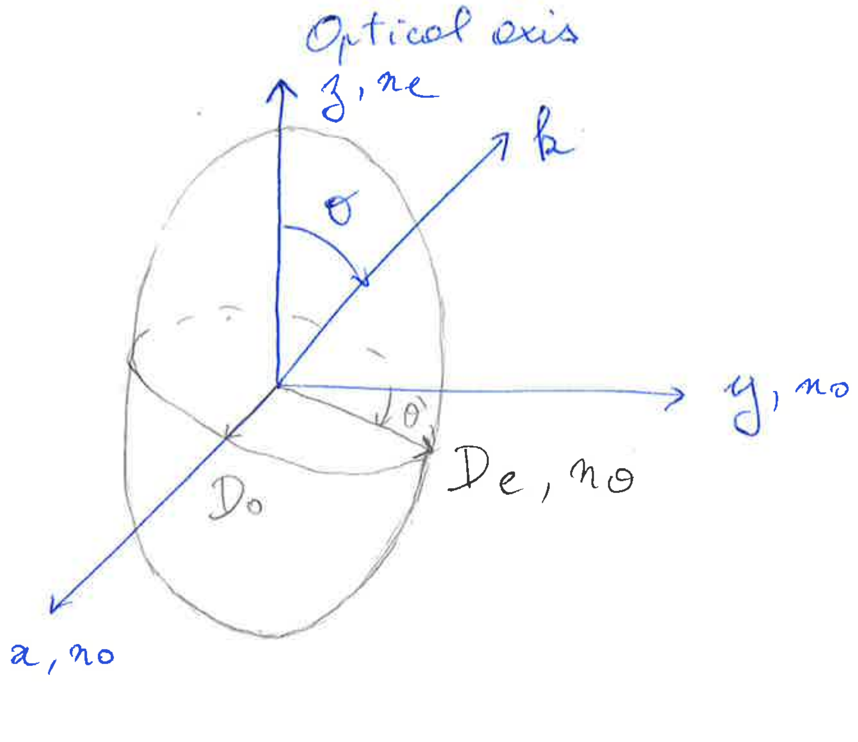
$D_e \nparallel E_e$

index n_θ given by

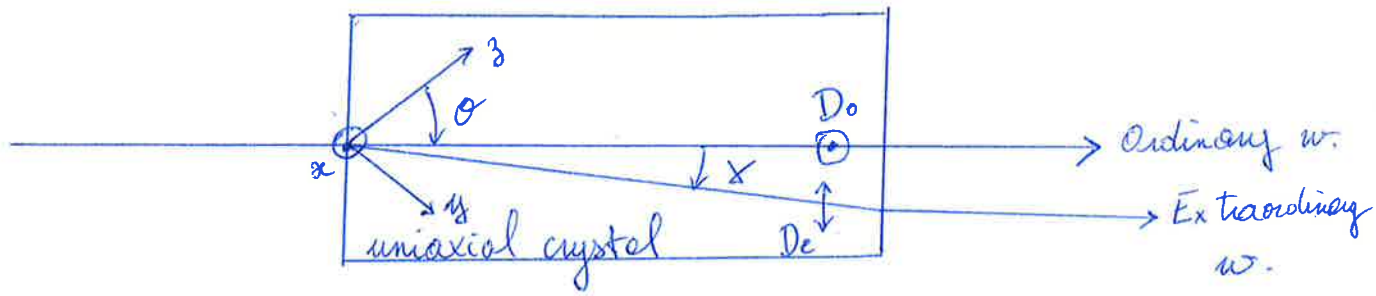
$$\frac{1}{n_\theta^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

Displaying polarization vectors using index ellipsoid formalism : (points M satisfying $OM = n \frac{D}{|D|}$

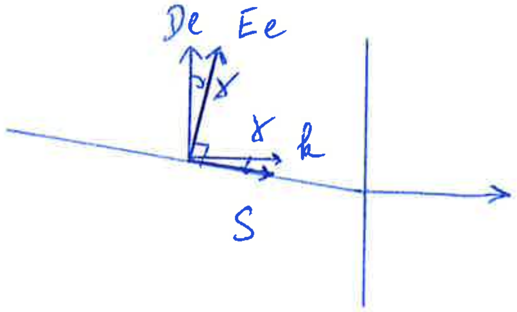
or $\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e} = 1$)

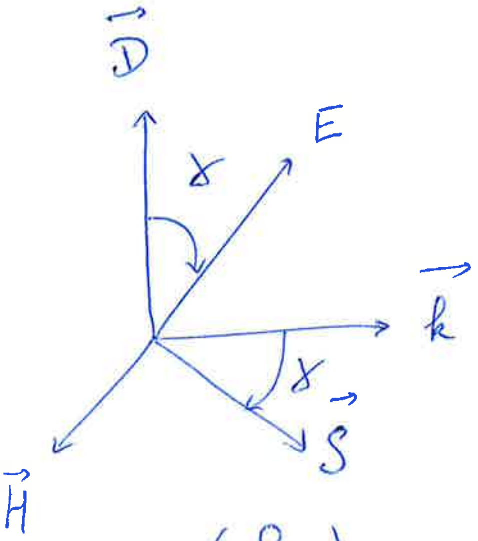


3)



For extraordinary wave





$$\tan\delta = \frac{|E \times D|}{|E \cdot D|} = \frac{\sin\delta}{\cos\delta}$$

$$\hat{e}\begin{pmatrix} 0 \\ e_y \\ e_z \end{pmatrix} \qquad E\begin{pmatrix} 0 \\ E_y \\ E_z \end{pmatrix}$$

$$D = \mathcal{E}_0 \underline{\underline{\mathcal{E}}}\, E = \begin{pmatrix} 0 \\ \mathcal{E}_0 n_o^2 E_y \\ \mathcal{E}_0 n_e^2 E_z \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$E = \begin{pmatrix} 0 \\ \cos\theta / n_o^2 \\ \sin\theta / n_e^2 \end{pmatrix}$$

(3f)

$$E \times D = \begin{pmatrix} -\dfrac{\sin\theta\cos\theta}{n_o^{\,2}} + \dfrac{\sin\theta\cos\theta}{n_e^{\,2}} \\ 0 \\ 0 \end{pmatrix}$$

$$E \cdot D = \frac{\cos^2\theta}{n_o^{\,2}} + \frac{\sin^2\theta}{n_e^{2}}$$

$$\tan\gamma = \frac{\dfrac{\sin 2\theta}{2}\left(\dfrac{1}{n_e^2} - \dfrac{1}{n_o^{\,2}}\right)}{\dfrac{\cos^2\theta}{n_o} + \dfrac{\sin^2\theta}{n_e}} = n_\theta^2 \, \frac{\sin 2\theta}{2}\left(\frac{1}{n_e^2} - \frac{1}{n_o^2}\right)$$

γ Walk-off angle results in double refraction, it vanishes for $\theta = \pi/2$.